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On Application of Improved Steepest Descent Method (ISDM) Algorithm to Determine the Optimal Control and Trajectories of Continuous-Time Linear Quadratic Regulator Problems (CLQRP)

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Abstract: **The Steepest Descent Method (SDM) is, appealingly, a straightforward gradient search technique, but not very effective in most applications, hence not recommended as a plug-and-play optimization procedure for several reasons. The SDM will often zigzag toward an optimum, requiring more steps of relatively little improvement toward the optimum. A significant disadvantage of the SDM for optimization problems is the number of iterations required to locate an extremum accurately. The search for a better, efficient, and robust technique, peculiar not to solving optimization problems alone but to solving optimal control problems, led to the Improved Steepest Descent Method (ISDM). The Improved Steepest Descent Method (ISDM) would solve nonlinear two-point boundary-value problems to determine the optimal controls and trajectories of Continuous-Time Linear Quadratic Regulator Problems (CLQRP). The method was employed to solve some CLQRP. The numerical results obtained from the ISDM show improvement over the classical SDM which is favourable compared with existing results.**

Keywords: **Optimal Control, Improved Steepest Descent Method, Continuous-Time Quadratic Linear Regulator Problem.**

1. INTRODUCTION

Classical variation techniques to find solutions to optimal control problems usually lead to nonlinear two-point boundary-value problems that cannot be solved analytically to obtain the optimal control law, or even an optimal open-loop control. In issues with plant dynamics and quadratic performance criteria (linear regulator and tracking systems), it has been found that it is possible to obtain the optimal control law by integrating numerically the Riccati type matrix differential equation.

An important class of optimal control problems called the Continuous-Time Linear Quadratic Regulator Problem (CLQRP) will be considered. The state equation time varying process by [1] to be controlled are described by:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0, \tag{1}$$

$$y(t) = C(t)x(t), \tag{2}$$

where $x(t)$ is indicates n-dimensional state vector, $u(t)$ represents the m-dimensional plant input control vector, and $y(t)$ stands for an r-dimensional output control vector. $A(t), B(t)$, and $C(t)$ are $n \times n$, $n \times m$, and $r \times n$ constant matrices, respectively, with $0 \leq r \leq m \leq n$, all specified.

According to [2], the plant input control vector that is mentioned here is a continuous-time, linear dynamical system with the following characteristics:

1. there is a time set $\{t_k\}$ such that $\{t_i\}$ = $(-\infty, \infty) = E_1$ for $T > t_0$, where t_0 is the initial time and T is the final or terminal time.

2. there is a set of states $\{x_i(t)\} = X = E_n$, is called the state space and E_n is referred to as the n-dimensional Euclidean vector space.

3. there is a set of controls or inputs $\{u_j(t)\} = U = E_m$, is referred to the control or input space, where E_m is the m-dimensional Euclidean vector space.

4. there is a function space Ω whose elements are bounded and measurable functions that map E_1 into U.

5. there is a set of outputs $\{y\} = Y = E_r$, that is called the output space.

If the control $u(.)$ is a given element of Ω and it is assumed that $x_u = \varphi(t;\ t_0, x_0, u(.))$ denote the solution of the system (1) starting from x_0 at time t_0 [i.e., $x(t_0) = x_0$], and this is generated by the control $u(.)$. Furthermore, let $y_u(t) = C(t)x_u(t)$ be the corresponding output trajectory suggested by [3]. Then, the CLQRP aims at evaluating the control $u(.)$, which then minimizes the quadratic performance measure.

More specifically, an explicit solution of (1), given an initial state, x_0, at t_0 and $t \geq t_0$, is as follows: (see [3])

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau \tag{3}$$

where $\Phi(t, \tau) = e^{A(t-\tau)}$ is the state transition matrix that is associated with the constant matrix $A(t)$. Similarly, the solution to (2), when an initial state, x_0, is given and the input, u(t) on $[t_0, \infty)$, is (see [4]):

$$x(t) = C(t)\Phi(t, t_0)x_0 + \int_{t_0}^{t} G(t, \tau)\, u(\tau)d\tau \tag{4}$$

where

$$G(t, \tau) = \begin{cases} C(t)\Phi(t, t_0)B(\tau); & for\ t \geq \tau \\ 0 ; & for\ t < \tau \end{cases} \tag{5}$$

It is the impulse response matrix.

Therefore, as mentioned in [5] and [6], the CLQRP for a linear dynamic system entails the determination of the optimal control $u^*(t)$, $t \in [t_0, T]$, that will minimize the quadratic performance index (see [4] and [2])

$$J(x, t_0, T, u(.)) = \frac{1}{2}x^T(T)Hx(T) + \frac{1}{2}\int_{t_0}^{T}\{x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\}dt \tag{6}$$

where H is a real symmetric positive semi-definite (non-zero) $n \times n$ matrix; the terminal state, $x_u(T) \in X$ is unconstrained, and the terminal time, T, may be either fixed prior or unspecified $(T > t_0)$. The superscript T denotes matrix transposition. $Q(t)$ is a real, $n \times n$ positive definite, and symmetric matrix. H and $Q(t)$ are both not identically zero. Since $R^{-1}(t)$ is positive definite, it possesses a unique positive definite square root, $R^{-1/2}(t)$. Similarly, the positive semi-definiteness of $Q(t)$ implies the existence of the unique positive semi-definite square root $Q^{-1/2}(t)$. The notation $H \geq 0$ indicates that the matrix H is positive semi-definite, that is, $x^T Hx \geq 0$.

Similarly, the notation $H > 0$ will indicate that H is positive definite. To minimize the performance index J, J must be finite, meaning it will become infinite if it is uncontrollable. The weighting matrices $Q(t)$ and $R(t)$ are selected by the control system designer to place bounds on the trajectory and control, respectively, while the matrix H and the terminal penalty cost $x^T(T)Hx(T)$ are included to ensure $x(t)$ stays close to zero near the terminal time.

From a design point of view, the control system designer may design the system such that the term $x^T Q(t)x(t)^*$ will be chosen to penalize deviations of the regulated state $x(t)$ from the expected equilibrium condition $x(t) = 0$, where the term $u^T R(t)u(t)$ is used to discourage the use of extremely huge control effort.

2. IMPROVED STEEPEST DESCENT METHOD (ISDM)

The discussion surrounding the Steepest Descent Method (SDM) as a variant of the gradient method will begin by focusing on its connectivity with the analogous calculus problem. According to Byron, (1973), one basic difficulty associated with the Steepest Accent/Descent Method is that, the amount of computation that is required to attain extremum a function depends largely on the sensitivity of the function to be able to change in each of the independent variables. By scaling the independent variables, it is often possible to change the value of certain partial derivatives then increasing the computational efficiency of the method. For functions that are not nicely scaled and that have non-zero off diagonal terms in the Hessian matrix, $H(x)$, that is corresponding to interactive terms such as (x_1, x_2) then, the negative gradient direction might not likely pass through the optimum directly as against the origin i. e. the optimum.

Oscillation is not the only convergence problem faced by the SDM. Ronald, (1998), concluded that, with small solution changes having a big objective function impact, numerical errors also can hopelessly bog down the procedure far from an optimal solution hence, more sophisticated technique is required to obtain a much more satisfactory and improved search algorithm. In the case of slimmer ellipses with a long narrow valley, Erwin, K., (1999), established that the convergence would be very poor and relatively slow.

From the investigation and conclusion of the authors in [7], that, in Steepest Descent Method (SDM), $p_i = -\nabla f(x_i)$, so, the gradients at points x_i and x_{i+1} are orthogonal. The orthogonality of successive search direction leads to a very inefficient zigzagging behaviour. Although, large steps are taken in early iterations, the step sizes shrink rapidly, and converging to an accurate solution of the optimization problem takes many iterations. It oscillates insignificantly as it approaches the stationary point that makes the method becomes relatively slow and unreliable to provide satisfactory results in many unconstrained nonlinear optimization applications. These and some other areas are what the Improved Steepest Descent Method (ISDM) is set out to tackle.

Let f be a function of two independent variables x_1 and x_2; the value of the function f at the point x_1 and x_2 is denoted by $f(x_1, x_2)$. One desires to determine the point x_1^* and x_2^* where the function f will attain its minimum values, $f(x_1^*, x_2^*)$.

According to [1], if the variables x_1 and x_2 are assumed not constrained by any boundaries, a necessary condition for x_1^*, x_2^* to be a point where f attains a relative minimum will be when the derivative of f vanishes at x_1^*, x_2^*, that is,

$$df(x_1^*, x_2^*) = [\frac{\partial f}{\partial x_1}(x_1^*, x_2^*)]\Delta x_1 + [\frac{\partial f}{\partial x_2}(x_1^*, x_2^*)]\Delta x_2 \quad (7)$$

This can be written much compactly as

$$df(x_1^*, x_2^*) \triangleq [\frac{\partial f}{\partial x}(x^*)]^T \Delta x = 0 \quad (8)$$

where $\frac{\partial f}{\partial x}$ is called the gradient of f with respect to x. Since x_1 and x_2 are independent, the components of Δx are independent, and (8) implies that

$$\frac{\partial f}{\partial x}(x^*) = 0. \quad (9)$$

However, for $f(x^*)$ to be a relative minimum of f, the gradient of f must be zero at the point x^* hence, (9) will stand for two different algebraic equations that are generally nonlinear such that, if the algebraic equations cannot be determined analytically for x^*, one possible approach is to visualize the minimization as a problem in hill climbing or steeping accent.

As opined by [1], the function f defines hills and valleys in the three-dimensional $(x_1, x_2, f(x_1, x_2))$ space. One classical method to determine the bottom of a valley is to pick a trial point $x^{(0)}$ and climb in a downward direction called steepest descent until a point x^* is reached where further movement in any direction will increase the value of the function.

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To make the climbing procedure efficient, according to [7], one chooses to climb in the direction of steepest descent, thus ensuring that the shortest distance is traveled in reaching the bottom of the hill. The steepest descent direction at $x^{(0)}$ is found as opined by [8] as estimating the gradient or derivative of the hill at the point $x^{(0)}$. The gradient vector will be normal to the elevation contour, and $z(x^{(0)})$ is the unit vector in the gradient direction at the point $z(x^{(0)})$; that is,

$$z\left(x^{(0)}\right) \triangleq \frac{\frac{\partial f}{\partial x}(x^{(0)})}{\left\|\frac{\partial f}{\partial x}(x^{(0)})\right\|} = \frac{\frac{\partial f}{\partial x}(x^{(0)})}{\sqrt{\left[\frac{\partial f}{\partial x_1}(x^{(0)})\right]^2+\left[\frac{\partial f}{\partial x_2}(x^{(0)})\right]^2}}. \tag{10}$$

When ascending the direction of $z\left(x^{(0)}\right)$ changes to the vector $-z(x^{(0)})$. The change in the value of x is given by

$$\Delta x \triangleq x^{(1)} - x^{(0)} = -\tau z(x^{(0)}), \tag{11}$$

and the step length, $\tau > 0$. With the selection of Δx, the differential, which is a linear approximation of the change in f, becomes

$$df\left(x^{(0)}\right) = -\tau \left[\frac{\partial f}{\partial x}\left(x^{(0)}\right)\right]^T z(x^{(0)}), \tag{12}$$

or by using (10),

$$df\left(x^{(0)}\right) = \frac{-\tau\left\{\left[\frac{\partial f}{\partial x_1}(x^{(0)})\right]^2+\left[\frac{\partial f}{\partial x_2}(x^{(0)})\right]^2\right\}}{\left\|\frac{\partial f}{\partial x}(x^{(0)})\right\|} \tag{13}$$

$$= \frac{-\tau\left\{\left[\frac{\partial f}{\partial x_1}(x^{(0)})\right]^2+\left[\frac{\partial f}{\partial x_2}(x^{(0)})\right]^2\right\}}{\sqrt{\left[\frac{\partial f}{\partial x_1}(x^{(0)})\right]^2+\left[\frac{\partial f}{\partial x_2}(x^{(0)})\right]^2}} = -\tau\left(\left[\frac{\partial f}{\partial x_1}\left(x^{(0)}\right)\right]^2 + \left[\frac{\partial f}{\partial x_2}\left(x^{(0)}\right)\right]^2\right)^{\frac{1}{2}} \tag{14}$$

Notice that this implies that

$$df\left(x^{(0)}\right) \le 0, \tag{15}$$

The equality of (15) will hold if $\frac{\partial f}{\partial x}$ is zero at $x^{(0)}$. This iterative procedure is continued by calculating $z(x^{(1)})$, the unit vector in the gradient direction at $x^{(1)}$, and employing the generalization (11) to determine the next point, $x^{(2)}$.

$$\Delta x \triangleq x^{(i+1)} - x^{(i)} = -\tau z(x^{(i)}) \tag{16}$$

A suitable value for the step length, τ, must also be chosen. If the value of τ is too large, one will overshoot the mark, and if τ is too small, too much time is spent trying to measure the slopes and the time spent climbing is not enough.

3. FUNCTION MINIMIZATION USING STEEPEST DESCENT

Suppose that the nominal control history $u^{(i)}(t)$, where $t \in [t_0, t_f]$, is specified and it is used to evaluate the state and the co-state differential equations

$$\dot{x}^{(i)}(t) = a(x^{(i)}(t), u^{(i)}(t), t) \tag{17}$$

$$\dot{p}^{(i)}(t) = -\frac{\partial \mathcal{H}}{\partial x}(x^{(i)}(t), u^{(i)}(t), p^{(i)}(t), t) \tag{18}$$

such that the nominal state and the corresponding co-state trajectory $x^{(i)}$, $p^{(i)}$ opined by [2] satisfies the boundary conditions

$$x^{(i)}(t_0) = x_0 \tag{19}$$

$$p^{(i)}(t_f) = \frac{\partial h}{\partial x}\left(x^{(i)}(t_f)\right). \tag{20}$$

Also, if the nominal control history satisfies

$$\frac{\partial \mathcal{H}}{\partial u}\left(x^{(i)}(t), u^{(i)}(t), p^{(i)}(t), t\right) = 0,\ t \in [t_0, t_f], \tag{21}$$

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then $x^{(i)}(t)$, $u^{(i)}(t)$, and $p^{(i)}(t)$ are tagged extremals. If the condition in (21) is not met, then the variation of the augmented functional J_a on the nominal state, co-state, and control history gives rise to

$$\delta J_a = [\frac{\partial h}{\partial x}\left(x^{(i)}(t_f)\right) - p^{(i)}(t_f)]^T \delta x(t_f)$$
$$+ \int_{t_0}^{t_f} \{[\dot{p}^{(i)}(t) + \frac{\partial \mathcal{H}}{\partial x}\left(x^{(i)}(t), u^{(i)}(t), p^{(i)}(t), t\right)]^T \delta x(t)$$
$$+ [\frac{\partial \mathcal{H}}{\partial u}\left(x^{(i)}(t), u^{(i)}(t), p^{(i)}(t), t\right)]^T \delta u(t)$$
$$+ a\left(x^{(i)}(t), u^{(i)}(t), t\right) - \dot{x}^{(i)}(t)]^T \delta p(t)\}dt, \quad (22)$$

With the following: $\delta x(t) \triangleq x^{(i+1)}(t) - x^{(i)}(t)$, $\delta u(t) \triangleq u^{(i+1)}(t) - u^{(i)}(t)$, and

$\delta p(t) \triangleq p^{(i+1)}(t) - p^{(i)}(t)$.

If (17) through (20) are satisfied, then

$$\delta J_a = \int_{t_0}^{t_f} \{[\frac{\partial \mathcal{H}}{\partial u}\left(x^{(i)}(t), u^{(i)}(t), p^{(i)}(t), t\right)]^T \delta u(t) dt \quad (23)$$

It must be recalled that (23) is the linear part of the control history increment is given by $\Delta J_a \triangleq J_a(u^{(i+1)}) - J_a(u^{(i)})$, (23a)

and if the norm given by

$$\|\delta u\| = \left\|u^{(i+1)}(t) - u^{(i)}(t)\right\|, \quad (24)$$

It is small, the sign, whether positive or negative, of ΔJ_a will be determined by the sign of δJ_a. However, the target is minimizing J_a, one will ultimately wish to make ΔJ_a negative.

If the change in u is chosen as

$$\delta u(t) = u^{(i+1)}(t) - u^{(i)}(t) = -\tau \frac{\partial \mathcal{H}^{(i)}}{\partial x}(\mathrm{t}), \qquad t \in [t_0, t_f], \quad (25)$$

and the value of τ is positive

$$\delta J_a = -\tau \int_{t_0}^{t_f} \left[\frac{\partial \mathcal{H}^{(i)}}{\partial u}(\mathrm{t})\right]^T \left[\frac{\partial \mathcal{H}^{(i)}}{\partial u}(\mathrm{t})\right] dt \leq 0, \quad (26)$$

This holds because the integrand in (26) is non-negative for all the values of $t \in [t_0, t_f]$. The equality holds on the ground that

$$\frac{\partial \mathcal{H}^{(i)}}{\partial u}(\mathrm{t}) = 0 \quad \text{for all} \quad t \in [t_0, t_f]. \quad (27)$$

The choice of δu such that $\|\delta u\|$ is sufficiently small imposes that each value of the performance measure will be at least as small as the preceding values [9]. Eventually, where the point the relative minimum value of J_a is attained, the Hamiltonian control vector $\frac{\partial \mathcal{H}}{\partial u}$ tends to zero through the entire time interval $[t_0, t_f]$. Sequel to these, one assumes that the state and co-state equations along with their corresponding boundary conditions in (17) to (20) are satisfied. On how to achieve this, this paper outlines the algorithm as it would be executed using a digital computer.

4. IMPROVED STEEPEST DESCENT METHOD (ISDM) ALGORITHM

It is important to outline the Steepest Descent Method algorithm for solving optimization problem as follows:

Step 1: Choose an initial or starting point x_0.

Step 2: Calculate (analytically or numerically) the partial derivatives of $f(x)$.

Step 3: Compute the search direction vector, $p_k = -\nabla f(x_k)$.

Step 4: Obtain the new variable value, x_{k+1}, using the relation $x_{k+1} = x_k - \alpha_k p_k$.

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Step 5: Test for convergence. If the convergence criterion is not met, then, return to step 3.

This process goes over and over until the convergence criterion is met. According to [10] and [11], the convergence criterion in practice could either be, $\|g_i\| \leq \varepsilon$, where ε is a chosen tolerance, then, terminate the sequence else, set $i = i + 1$ and return to step 3 and terminate the process when $f(x_{i+1}) = \mathrm{f}(x_i)$.

This procedure is only adaptable to optimization problems hence, the Improved Steepest Descent Method (ISDM) algorithm follow in what is next. According to [1], the optimal control problems (1) and (6) may be solved with the ISDM via the following steps:

Step 1: Let the iteration index i be zero. Choose an arbitrary discrete approximated value for the nominal control history $u^{(0)}(t), t \in [t_0, t_f]$. The value is stored on the memory of the computer. This is achieved by subdividing the time interval $[t_0, t_f]$ into N subintervals (generally of equal duration) and considering the control $u^{(0)}$ as being piecewise-constant during each of these subintervals; that is

$$u^{(0)}(t) = u^{(0)}(t_k), t \in [t_k, t_{k+1}), k = 0, 1, 2, \dots, N-1. \quad (28)$$

Step 2: Integrate the state equations (17) and (18) from t_0 to t_f with initial conditions $x(t_0) = x_0$ using the nominal control history $u^{(i)}$ and store the resulting state trajectory $x^{(i)}$ as a piecewise-constant vector function.

Step 3: Calculate $p^{(i)}(t_f)$ and substitute $x^{(i)}(t_f)$ from Step 2 into (20). Using the value of $p^{(i)}(t_f)$ as the initial condition and the piecewise-constant values of $x^{(i)}$ stored in Step 2, integrate the co-state equations from t_f to t_0 hence, evaluating $\frac{\partial \mathcal{H}^{(i)}(t)}{\partial u}$, $t \in [t_0, t_f]$, and store this function in piecewise-constant fashion.

Step 4: Test for convergence. If γ is a preselected positive constant such that

$$\left\| \frac{\partial \mathcal{H}^{(i)}}{\partial u} \right\| \leq \gamma, \quad (29)$$

and

$$\left\| \frac{\partial \mathcal{H}^{(i)}}{\partial u} \right\|^2 \triangleq \int_{t_0}^{t_f} \left[\frac{\partial \mathcal{H}^{(i)}}{\partial u}(t) \right]^T \left[\frac{\partial \mathcal{H}^{(i)}}{\partial u}(t) \right] dt, \quad (30)$$

stop the iterative process, and output the values of the state and control extremals. If the stopping criterion (29) is not satisfied then, according to [12], there will be need to generate a new piecewise-constant function given by

$$u^{(i+1)}(t_k) = u^{(i)}(t_k) - \tau \frac{\partial \mathcal{H}^{(i)}}{\partial u}(t_k), k = 0, 1, \dots, N-1, \quad (31)$$

where $u^{(i)}(t) = u^{(i)}(t_k)$ for $t \in [t_k, t_{k+1})$, $k = 0, 1, \dots, N-1$. (32)

Then, replace $u^{(i)}(t_k)$ with $u^{(i+1)}(t_k), k = 0, 1, \dots, N-1$, and return to Step 2.

The value of the predetermined terminating constant γ will depend on the problem to be solved and the desired accuracy of the solution as observed by [13]. The step length, τ, is generally determined by some ad-hoc strategy. One possible strategy is to choose a value for τ that will influence the value of ΔJ_a.

From (31), one observes that

$$\delta J_a = -\tau \left\| \frac{\partial \mathcal{H}^{(i)}}{\partial u} \right\|^2 \leq 0. \quad (33)$$

To effect an approximate change of q percent in J_a, one select τ as

$$\tau = \frac{\frac{q}{100}|J_a|}{\left\| \frac{\partial \mathcal{H}^{(i)}}{\partial u} \right\|^2}. \quad (34)$$

An alternative strategy for selecting τ is to use a single variable search. An arbitrary starting value of τ will be chosen to compute $\frac{\partial \mathcal{H}^{(i)}}{\partial u}$ and to evaluate $u^{(i+1)}$ using (21).

The ISDM will now be employed to solve some benchmark CLQRP, as shown in the next section.

5. COMPUTATIONAL RESULTS

The following CLQRP will be solved using the ISDM algorithm described above.

Problem 1: What is the optimal trajectory and control for the system

$\dot{x}(t) = -3x(t) + 2u(t); 0 \leq t \leq 1,$

$x(0) = 2$ and $u_0(t) = 1.0$, that will minimize the performance index

$J = x^2(1) + \frac{1}{2}\int_0^1 \{x^2(t) + u^2(t)\}\, dt$

Table 1: Computational Results for Solution of Problem P1.

Iteration	X(t)	U(t)	P(t)	$\frac{\partial \mathcal{H}}{\partial u}$
0	2	1.0	-	-
1	0.73304942	0.31356046	2.9321977	6.8643954
2	0.23512918	0.94101067e-1	0.94051674	2.19459394
3	0.71317093e-1	0.27637286e-1	0.28526837	0.66463781
4	0.02105820	0.80269919e-2	0.84232826e-1	0.19610294
5	0.61333273e-2	0.23176308e-2	0.02453330	0.57093611e-1
6	0.17735222e-2	0.66704994e-3	0.70940889e-2	0.01650580
7	0.51085812e-3	0.19165845e-3	0.20434325e-2	0.47539149e-2

Problem 2: Find the optimal trajectory and control for the system

$\dot{x}(t) = -x(t) + u(t); 0 \leq t \leq 1, x(0) = 4$ and $u_0(t) = 1.0$, that minimizes the performance measure

$J = x^2(1) + \int_0^1 \frac{1}{2} u^2(t)\, dt$

Table 2: Computational Results for Solution of Problem 2.

Iteration	$x(t)$	$u(t)$	$P(t)$	$\frac{\partial \mathcal{H}}{\partial u}$
0	4	1	-	-
1	2.10363832	0.58544671e-1	8.41455329	9.41455329
2	0.81089258	-0.27166683	3.24357032	3.30211499
3	0.12658452	-0.29513395	0.50633809	0.23467126
4	-0.13999246	-0.20962363	-0.55996959	-0.85510354

Problem 3: Minimize $J = x^2(1) + \frac{1}{2}\int_0^1 [4x^2(t) + 2u^2(t)]\, dt$

Subject to the constraint

$\dot{x}(t) = -x(t) + u(t); 0 \leq t \leq 1, \ x(0) = 3$ and $u_0(t) = 1.0.$

Table 3: Computational Results for Solution of Problem 3.

Iteration	$x(t)$	$u(t)$	$P(t)$	$\frac{\partial \mathcal{H}}{\partial u}$
0	3	1	-	-
1	1.73575888	-0.48860711	6.94303553	14.8860711
2	0.32969141	-0.70349952	1.31876564	2.14892418
3	-0.32340982	-0.37442172	-1.29363928	-3.29077809

Problem 4: Minimize J $= \frac{1}{2}x^2(0.04) + \frac{1}{4}\int_0^{0.04}[x^2(t) + u^2(t)]\,dt$

Subject to the constraint

$\dot{x}(t) = -10x(t) + u(t); 0 \le t \le 0.04, \quad x(0) = 2$ and $u_0(t) = 1.0.$

Table 4: Computational Results for Solution of Problem 4.

Iteration	$x(t)$	$u(t)$	$P(t)$	$\frac{\partial \mathcal{H}}{\partial u}$
0	2	1	-	-
1	1.37360809	0.76263919	2.74721618	2.37360809
2	0.94589972	0.59178533	1.89179944	1.70853891
3	0.65356552	0.46725022	1.30713104	1.24535082
4	0.45350237	0.37517496	0.90700475	0.92075259
5	0.31636053	0.30602141	0.63272151	0.69153546
6	0.22215172	0.25320414	0.44430339	0.52817311
7	0.15726037	0.21215766	0.31452073	0.41046447
8	0.11240919	0.17970097	0.22481838	0.32456684
9	0.81274513e-1	0.15360342	0.16254903	0.26097548
10	0.59543933e-1	0.13228869	0.11908787	0.21314736
11	0.44274784e-1	0.11463234	0.88549569e-1	0.17656347
12	0.33457474e-1	0.99823359e-1	0.66914948e-1	0.14808981
13	0.25718192e-1	0.87269203e-1	0.51436383e-1	0.12554155
14	0.20116551e-1	0.76530632e-1	0.40238302e-1	0.10738571
15	0.16007561e-1	0.67276813e-1	0.32015123e-1	0.92538194e-1
16	0.12948171e-1	0.59254314e-1	0.25896342e-1	0.80224984e-1
17	0.10632915e-1	0.52265592e-1	0.21265829e-1	0.69887229e-1
18	0.88505475e-2	0.46153978e-1	0.17701095e-1	0.61116139e-1

Problem 5: Minimize J $= 10{x_1}^2(T) + \frac{1}{2}\int_0^T[{x_1}^2(t) + 2{x_2}^2(t) + u^2(t)]\,dt$

$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u, x(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \; u_0(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and the final time $T = 10$.

6. CONCLUSION

Computationally, the ISDM was tested on some continuous-time linear-quadratic regulator problems, and the results obtained in each case showed improvement over classical methods hence, it compares favourably with other numerical techniques. Our numerical results for these problems are presented in Tables 1 through 5. From the tables, Problem 1 is seen to readily converge at the seventh iteration when compared with the analytical result, $x^* = 0.00017736422$ and $u^* = 0.00019165842$. Problem 2 and Problem 3 are seen to converge at the fourth and third iterations, respectively, when compared with the existing analytical results, the state and control values are gotten as $x^* = -0.14998465235$, $u^* = -0.2735776132$, and $x^* = -0.3237098216$, $u^* = -0.3742215284$ respectively. While Problem 4 converged at the 18th iteration to $x^* = 0.008850547533$ and $u^* = 0.04615397268$.

Moreover, the terminating criterion was fixed at $\left\| \frac{\partial \mathcal{H}^{(i)}}{\partial u} \right\| \le 10^{-10}$. The terminating criterion is seen to be too relaxed, so that one could substantiate the usage of the ISDM in solving this class of problem. Going by these results, it becomes clear that, on determining the optimal controls and trajectories of continuous-time linear-quadratic regulator problems using iterative numerical techniques, the Improved Steepest Descent Method (ISDM) is said to stand out, relevant hence, by extension aside curtailing its usage to optimization problems alone it is hereby recommended for use in solving CLQRP.

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